

Dipolar interaction and the Manning formula

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ABSTRACT

In this work we want to show that the mathematical model of quantum mechanics, led to its classical approach, is able to reproduce as close macroscopic experimental results captured by the Manning formula, sufficiently verified through their diverse applications in hydraulics. Molecular interaction between the fluid and the wall of the vessel is studied decomposing the Hamiltonian in two parts: free, and interacting. Scaling process is considered from molecular to hydraulic. Participation of the symmetries of Saint-Venant equation in the hydraulic gradient is taken into account. Correlations between different variables are set. The magnitude of scale change is estimated. We conclude that the Compton wavelength induces to the Boussinesq viscosity concept and the characteristic length of the viscous sublayer.

Keywords – Dipolar interaction, Manning formula, Navier-Stokes fractional equation, Compton wavelength, Compton wavelength

I. INTRODUCTION

The purpose of this paper is to show that the mathematical model of quantum mechanics led to its classic approach is able to reproduce as close macroscopic experimental results collected in the Manning formula, sufficiently verified through its wide use in irrigation channels, rivers and in many different hydraulic applications.

According to the references quoted, we know that the Navier-Stokes fractional equation is supported by the Darcy's fractional law, which simplified in the boundary layer, and based on their thinness, allows to define a multifractal, which has as resolution the reciprocal of the Reynolds number and as dimension the Blasius exponent, which depends on the spatial occupation rate and the pressure gradient exponent according to Falkner-Skan approximation, [1, 2].

When we calculate the friction force between the wall and the fluid, its representation as a non-local operator in the form of a fractional derivative is obtained; being an implication of the Darcy fractional law, it shows consistency with the viscous force. Subsequently, an expression for the hydraulic gradient is obtained, with two options: one for smooth surface and the other for rough surface, in which both the exponent for Reynolds and Froude numbers, depend on the Blasius exponent again.

Moreover, based on the molecular scale we can describe the interaction between the fluid and the wall as a dipole interaction between the two materials. According to quantum mechanics as a superposition of two free particles and the interaction, Hamiltonian is described. Rellich

theorem is use to expand in a power series an Hermitian operator which depends on a parameter, the Planck constant, and according to the perturbation, energy associated with the interaction between dipoles is obtained, which is the potential form of the Van der Waals forces, or energy associated with them. It contains two constants: Planck and fine structure.

Interaction is dimensionless, using the Compton wavelength as reference length, and also the fine structure constant as dimensionless form of the square of the elementary charge. The shape resembles that obtained for the hydraulic gradient, and might be thought as a Lie derivative of a certain magnitude, the dimensionless interaction energy, which is proportional to itself.

But it also has a relatively large magnitude that is associated with the contribution of relativity and with the constant structure, and other small magnitude associated with the Planck constant; if the method of generalized Cantor is apply, a way to scale from the molecular microscopic to the semi-classical macroscopic may be reach, and the hydraulic gradient could be obtained. At that time the goal is to find a link between the rate of spatial occupation, and fundamental constants of the quantum mechanics, by relating the Blasius exponent with the micro-world.

The main hypothesis is that the mathematical model of quantum mechanics contains that's of the classical mechanics, but not vice versa, so the frictional interaction from the molecular scale, according to quantum mechanics, scale to the macro

level of fluids, while a relativistic expression is scale to the classic, must produce a result close to the Manning formula.

II. PROBLEM FORMULATION AND SOLUTION OF THE MODEL

In the quantum level, two atoms with an external electron and a distance of the order of 10 \AA (Angstrom) are assume, which is already a first approximation by simplifying the exchange interaction based on the interchangeability of the electrons. The Hamiltonian of the system, equation (1), correspond to atom 1, plus atom 2, plus the interaction energy that is denoted by $W(r_1, r_2, R)$, where $R = |r_2 - r_1|$ is the distance between the two atoms;

$$H = H_1 + H_2 + W(r_1, r_2, R) \quad (1)$$

W is the energy associated with the interaction between dipoles, and in his classic description is represented by equation (2), where e^2 is the square of the elementary charge. According to the mathematical model of quantum mechanics, we do that the classical energy matches an interaction or disturbance operator. After the calculation of the angular part and introducing the notation w in the result, we have (2):

$$W = \frac{e^2}{R} \left(r_1 \cdot r_2 - 3 \frac{(r_1 \cdot R)(r_2 \cdot R)}{R^2} \right) = \frac{e^2}{R} w \quad (2)$$

Steady state that corresponds to two isolated atoms or without interaction is considered, plus the disturbing that it introduces. We assumed that the interaction is weak. Solutions of the stationary Schrödinger equation are sought, so we try to find the eigenvalues and the corresponding eigenfunctions of the perturbed operator, under the assumption that the eigenvalues are non-degenerate and the spectrum is discrete.

With the method of successive approximations, we denote $\{\psi_m^0\}_m$ as a Schauder basis or complete set of eigenfunctions of the operator H^0 . By density, each state ψ supports a representation of the form $\psi = \sum_m c_m \psi_m^0$. So 1st order correction of

the fundamental is $c_k^{(1)} = \frac{W_{kn}}{E_n^0 - E_k^0}$, $k \neq n$,

where W_{kn} are the elements of the matrix and E_k^n

are the energy eigenvalues. For the 2nd order

correction results $E_n^{(2)} = -\sum_{k \neq n} \frac{|W_{kn}|^2}{E_n^0 - E_k^0}$, [3].

Subsequently, it is shown that for the two hydrogen atoms in the fundamental state, and based on the Hermitian, the coefficient in the expression of the interaction is found to be $6e^2 a_0^5$, where (e, a_0)

is the elementary charge, $(a_0 = \hbar / me^2)$ the Bohr radius and (\hbar, m) the Planck's constant and the mass of the outer electron, respectively.

Finally, the energy is obtained by means of (3), wherein the fine structure constant is $\alpha = e^2 / \hbar c$, the Compton wavelength is $\tilde{\lambda} = \hbar / mc$, and c is the universal constant of the light velocity,

$$W(R) = -\frac{6e^2 a_0^5}{R^6}, \quad (3)$$

$$W(R) = -6(\alpha^{-1})^4 \frac{mc^2}{(R/\tilde{\lambda})^6}$$

2.1 MULTISCALE ANALYSIS

This is similar to the decomposition of Reynolds but reiterated. The idea is to divide a signal into two parts, as a direct sum, so that the first contains the more regular part of the signal, while the second, contains the more irregular, or in low frequency and high frequency, [4, 5].

According to the projection theorem, the total space can be decomposed into a closed subspace and its orthogonal complement where the sum is direct. Therefore, for this breakdown in the two sub-spaces should be considered the two projections. Then the procedure on the regular component is repeated, so that the original is now represented in three parts, being the last term in a different, previous or higher resolution scale.

Thus it can be continued with a third stage in a similar way, but, the orthogonal complement contains the superposition of the preceding scales with a coarse resolution. So, at the nth step, the orthogonal complement contains the superposition of all immediately preceding scales, with a coarser resolution than the current, the nth resolution.

In order N , we denote the sub-spaces of low and high frequencies as the pair (V_N, V_N^\perp) , so that the relationship between the original scale of order 0 and that of order N is represented by (4):

$$f(2^N x) \in V_0 \Leftrightarrow f(x) \in V_N \quad (4)$$

Consider now the sub-space of the N -th order as that in which the dipolar interaction occurs, so if it's denoted by $W(m/m_0, R/\lambda)$, then $W(2^N(m/m_0, R/\lambda))$ should be in the macro sub-space of the hydraulic. Reciprocally, when it is decomposed to the more and more fine resolutions, these will contain from the dipolar interactions and quadrupole onwards, while it will produce orthogonal complements with the coarser resolutions.

2.2 THE SYMMETRIES

We have assumed that V_0 denotes the hydraulic macro space; however it is obtained from a projection of the Navier-Stokes fractional equation space, constructed through an approximation. So if we compose the projection from space V_N with an uprising from V_0 to the space of the Navier-Stokes equation, we will have an application from the bipolar interaction space until the energy space of the Navier-Stokes equation.

The space where the Navier-Stokes fractional equation is performed is called the energy space of Sobolev $(H_0^1(\Omega))^d$, obtained through the closure of differentiable functions of any order with compact support, and under the condition of non-divergence, whose closed subspace, denoted H_{div} , is obtained by adding the non-divergence condition again, [6].

A projection is made through an approximation. The V_0 space, where Saint-Venant equation is performed, is obtained. So if we compose the projection from the space V_N with an uprising from V_0 to the space of the Navier-Stokes equation, we have the said application from the bipolar interaction space to the energy space of the Navier-Stokes equation H_{div} .

In reference [7], an equation of closure is exposed, which is obtained by the method of inverse problems based on a group analysis of the equations system and the introduction of prior information, that show experimental results, both in a qualitative and quantitative way.

In the cited reference [7] the hydraulic gradient is obtained as an inverse problem, and it is shown that possesses the property of form invariance, and is stated as follows:

The Lie derivative of the friction slope in the direction of the infinitesimal generator of group \mathbf{V} , is proportional to the same slope, $L_{\mathbf{V}}J = c(x, t)J$, being the generator field variables, (U, h) : the

average velocity and the water depth. So that the friction slope may be find in different ways but by the formula $J = cte \frac{U^a}{h^b}$.

And for both hydraulically possible states the following results are obtained: for smooth

$$J = cte \frac{(U/U_*)^2}{(U_*R_h/v_2)^\theta}, \quad \text{and} \quad \text{roughness}$$

$$J = cte \frac{(U/\sqrt{gR_h})^{2-\theta}}{(R_h/y_r)^{(b-1+\theta/2)}}, \quad \text{being}$$

$(U, U_*, R_h, g, v_2, y_r)$ the average velocity, the shear velocity, the hydraulic radius, gravity, kinematic viscosity and the roughness height, respectively; θ is the Blasius exponent and b is the exponent from the hydraulic radius, in the flow resistance formulas, [1].

Next step is the identification from where could start the cascade process. It is intended to identify the source of the cascade.

2.3 IDENTIFICATION

Correlations that we propose are based on the infinitesimal generator in the Lie derivative $L_{\mathbf{V}}$ which, as stated above, depends on two field variables: the first is the velocity and is therefore related to the kinetic energy in the Hamilton operator, the second is the water depth, and is related to the potential energy in the Hamiltonian.

The expression of the interaction energy through the rest energy of the electron is dimensionless, as shown in (5),

$$\frac{W(R)}{m_0c^2} = -6(\alpha^{-1})^4 \left(\frac{m}{m_0} \right) \frac{1}{(R/\lambda)^6},$$

$$\frac{m}{m_0} \approx 1 + \frac{1}{2} \left(\frac{v}{c} \right)^2, \quad (5)$$

The problem is decomposed into two classes according to the two possible regimes; so, for the hydraulically smooth must be (6):

$$6(\alpha^{-1})^4 (m/m_0) \mapsto (U/U_*)^2, \quad (R/\lambda)^6$$

$$\mapsto (R_h/l_c)^\theta, \quad l_c = v_2/U_* \quad (6)$$

While for hydraulically rough is (7):

$$6(\alpha^{-1})^4 (m/m_0) \mapsto (U/\sqrt{gR_h})^{2-\theta}, \quad (R/\lambda)^6$$

$$\mapsto (R_h/y_r)^{1-b-\theta/2} \quad (7)$$

Consequently, correlations proposed allow to obtain the scalings (8):

$$2^{N_1} (R/\tilde{\lambda})^\theta = (R_h/l_c)^\theta, \quad (8)$$

$$2^{N_2} 6(\alpha^{-1})^4 (m/m_0) = (U/U_*)^2$$

And therefore, the ratio of total scale between the macro and the micro will be (9):

$$2^{(N_2-N_1)} \left(6(\alpha^{-1})^4 \right) = 6.022 \times 10^{23}, \quad (9)$$

$$N = N_2 - N_1 = 48.017$$

And it is found that the transition from quantum to classical is based on the obvious inequality $\lambda_c \ll y_r$ (the Compton wavelength and the roughness height).

Then, as a consequence, we infer that the exponent of Blasius comes to terms with the microworld through the Compton wavelength, (10):

$$2^{N_1} (R/\tilde{\lambda})^\theta = (R_h/l_c)^\theta, \quad \theta = \frac{\ln(2^{N_1} (R/\tilde{\lambda}))}{\ln(R_h/l_c)} \quad (10)$$

But in addition and similarly to the multi-scaled process, Manning formula is recovered, which is widely used to estimate flow resistance in channels, in hydraulically rough and turbulent flow regime, by means of the following correlations:

$\left(\frac{g}{(1.49)^2}, b = 4/3, n \right)$ being the dimensionless constant, the exponent of the hydraulic radius and Manning roughness, and in the limit $\theta \rightarrow 0$, representing the turbulent state, fully developed, [8]:

$$6(\alpha^{-1})^4 \left(\frac{\tilde{\lambda}}{R} \right)^6 \left(\frac{m}{m_0} \right) \xrightarrow{\theta \rightarrow 0, b=4/3} \frac{g}{(1.49)^2} \left(\frac{n^6}{R_h} \right)^{1/3} \left(\frac{U}{\sqrt{gR_h}} \right)^2 \quad (11)$$

Moreover, the kinematic viscosity can be represented in accordance with the concept of Boussinesq, in the form

$$\nu_2 = U_* l_2, \quad l_2 = \left(\frac{\eta^4}{\kappa \lambda_T} \right)^{1/3}, \quad \text{where } (\eta, \kappa, \lambda_T)$$

denote: Kolmogorov length, Kármán constant and the Taylor integral scale, this last assumed as the scale of vortices that contains the energy and where the Kolmogorov cascade is initiated, [9].

Furthermore and accordingly, since the viscosity is $\nu_\alpha = U_* l_\alpha^\beta$, the kinematic viscosity can be scaled as is shown in (12), and in order to be used in natural flow channels with sediments:

$$\nu_2 = \nu_2(\phi)^{-\beta}, \quad \phi = \frac{l_2^{1/\beta} l^{1-1/\beta}}{l_\alpha} < 1, \quad (12)$$

III. DISCUSSION

Is sufficiently justified the scaling process between an Hilbert space of energy, and one of its closed subspaces, then repeat it on subsequent decompositions of the same type, through the Projection theorem, as shown by the multiple applications of the wavelet transform, where in each stage of the process a signal is decomposed into a trend plus fluctuations [5]; and therefore, Reynolds decomposition is only a reference.

Indeed, multiscale scaling process gradually shows details that are initially hidden or encrypted and will reveal details gradually or particularities or singularities not observed in the initial space. Moreover, the multiscale process goes in both directions, from the micro to the macro as in the opposite direction, as shown in (4). However, the model is not unique, because we are not taking into account from the quadrupole interactions forward; and in counterpart, we are not considering the subspaces complementary to close subspace.

IV. CONCLUSION

As shown in (5), the change of scale from the micro to the macro is determined by two important constants: Avogadro's number and the Sommerfeld fine structure constant. The micro phenomenon is quantum and relativistic, the first measured by the Compton wavelength, the second by the relativistic energy of the electron. As a corollary to equation (11) it is seen that the concept of Boussinesq viscosity is induced by the Compton wavelength. Multiscale process, the intermediate control of the hydraulic gradient and the constant adjustment reproduce the Manning formula. With regard to future applications it is observed that the kinematic viscosity can be rescaled by appropriately small solid particles.

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